STABILITY AND SYSTEM TIME DISTRIBUTION OF DYNAMIC TRAVELING REPAIRMAN PROBLEM

JIANGCHUAN HUANG† AND RAJA SENGUPTA‡

Abstract. A good model for the dynamic vehicle routing problem is the Dynamic Traveling Repairman Problem (DTRP) [4]. The DTRP literature has focused on optimizing the expected value of system time, defined as the elapsed time between the arrival and the completion of each task. We focus on the stability and distribution of system time, including its variance. This paper establishes a partially policy independent necessary and sufficient condition for stability in the DTRP. The policy class includes some of the policies proven to be optimal for system time expectation under light and heavy loads in the literature. We propose a new policy named PART-n-TSP and compute a good approximation for its system time distribution. PART-n-TSP has lower system time variance than PART-TSP [31] and Nearest Neighbor [4] when the load is neither too small or too large. We prove that PART-n-TSP is also optimal for system time expectation under light and heavy loads.

Key words. Dynamic Traveling Repairman Problem, Polling Systems, Economy of Scale

1. Introduction. Dynamic vehicle routing problems arise when one needs to serve tasks that arrive in time and space. The objective is to schedule the tasks in an economic way to achieve good service level according to some performance metric, e.g., average waiting time, throughput, delivery probability of the tasks, or the total distance traveled by the moving servers (vehicles). There are many practical application where the dynamic vehicle routing problems arise. For example, (i) Google Street View, Google is running numerous data-collection vehicles with mounted cameras, lasers, a GPS and several computers to collect street views while minimizing the distance travelled [2]. (ii) Real-time traffic reporting, a radio station uses a helicopter to overfly accident scenes and other areas of high traffic volume for real-time traffic information. (iii) Unmanned aerial vehicle (UAV)-based sensing, a fleet of UAVs equipped with greenhouse gas sensors collect airborne measurements of greenhouse gases at several sites in California and Nevada, executing every task at its location before its deadline [18]. (iv) Enabling mobility in wireless sensor networks (WSN), mobile elements (vehicles) capable of short-range communications collect data from nearby sensor nodes as they approach on a schedule [23].

The Dynamic Traveling Repairman Problem (DTRP) [4] is a good way to model the dynamic vehicle routing problem: A convex region \( A \) of area \( A \) contains a vehicle (server) that travels at constant speed \( v \). Tasks arrive according to a Poisson process with rate \( \lambda \). Each task \( i \) is located at \( X_i \in A \), and has size \( B_i \). \( X_i \) is independent and identically distributed (i.i.d.) with probability density function (pdf) \( f_X(x), x \in A \). \( B_i \) is i.i.d. with pdf \( f_B(s), s \in [0, \infty) \). \( E[B_i] = b \), which is assumed to be finite. Define load \( \rho = \lambda b \). The system time of task \( i \), denoted \( T_i \), is defined as the elapsed time between the arrival of task \( i \) and the time task \( i \) is completed. It is a measure of system performance. An earlier formulation similar to the DTRP can be found in [10].

The DTRP resembles an M/G/1 queue in the time dimension but looks like a vehicle routing problem in the space dimension. As we know in queuing theory,
\( \rho = \lambda b < 1 \) is a necessary and sufficient condition for all work conserving M/G/1 queues [9, sec. II.4.2]. However, there is no such policy-independent stability condition for the DTRP, which seems to be a “spatial version” of the M/G/1 queue. The known stability conditions for the DTRP are policy-dependent [4, 31, 20].

This paper makes progress towards finding stability conditions for the DTRP that are less policy dependent than those in the literature. We establish \( \rho + \lambda b_d < 1 \) as a necessary and sufficient stability condition for the class of Polling-Sequencing (P-S) policies satisfying unlimited-polling and economy of scale in Theorem 2.12. This stability condition is identical to the necessary condition for stability given in [5]. We show that important policies for the DTRP in the literature fall in the P-S class and satisfy the two properties. The extra term \( b_d \) is the limit of mean travel time as the number of tasks in a polling station goes to infinity. We prove that the existence of \( b_d \) is a consequence of economy of scale. \( b_d \) is policy-dependent, but it only depends on the sequencing phase of the P-S policy. Since the value of \( b_d \) can be derived in the static setting, we do not need to analyze or simulate the dynamic queueing system of DTRP to get \( b_d \). We only need to analyze or simulate to obtain the statistics of sequencing \( N \) tasks, where \( N \) is a random variable, and the task locations are distributed in some fashion.

Our second contribution to the DTRP is the distribution of system time \( T \), defined as the elapsed time between the arrival and the completion of each task. Knowing the distribution of the system time \( T \), together with its expectation \( E[T] \) and variance \( Var[T] \) or standard deviation \( \sigma[T] \), enables the expectation-variance analysis of the system under uncertainties [12, 22]. On entering a McDonalds, one may ask not just “What is my expected service time?” but also “How certain is this value?” We show in Tables 3.1 and 3.2 in Section 3 that two policies at the same load level can be incomparable in the sense that one has low expectation of system time but high variance while the other has high expectation of system time but low variance. In practice, highly variable system time can be even more frustrating than large mean system times [13, 17]. The literature discusses the distribution of system time \( T \), together with its expectation and variance, for the FCFS policy and its variations such as the SQM and partitioning-FCFS [4]. This is in sharp contrast to queueing theory where the distribution of the system time or its moments are known for a wide variety of policies. See for example [26, 30]. To illustrate our point, the expected system time of the FCFS, SQM and partitioning-FCFS policy is not as good as NN [4] and PART-TSP [31] or DC [20] at most load levels.

In Section 3, we propose a policy in the P-S class called the PART-\( n \)-TSP policy. We give a good approximation for the distribution of the system time that is easy to compute. We do this by utilizing approximation results for the distribution of system time \( T \), together with \( E[T] \) and \( Var[T] \) known for polling systems [8, 15]. Figure 3.5 shows that the cumulative distribution function (cdf) of the system time as computed by our method is very close to the cdf of the system time as obtained by Monte-Carlo simulation. We show that FCFS, partitioning-FCFS and \( n \)-TSP [4] are special cases of PART-\( n \)-TSP, meaning PART-\( n \)-TSP can be optimized to have better performance than the three. We also compare PART-\( n \)-TSP with PART-TSP [31] and Nearest Neighbor [4] on \( E[T] \) and \( \sigma[T] \) in Tables 3.1 and 3.2, since the latter two are considered near optimal in the literature. The \( E[T] \) and \( \sigma[T] \) under PART-\( n \)-TSP are obtained by our approximation. The \( E[T] \) and \( \sigma[T] \) under PART-TSP and NN are obtained by simulation. The results show that NN achieves lower \( E[T] \) than both PART-\( n \)-TSP and PART-TSP for all loads \( \rho \in \{0.1 \ldots 0.9\} \) simulated by us. PART-
n-TSP achieves lower $E[T]$ than PART-TSP when $\rho$ is not too small or too large, e.g. when $\rho \in \{0.3, \ldots, 0.7\}$. Also, PART-$n$-TSP achieves lower $\sigma[T]$ than PART-TSP and NN when $\rho$ is not too small or too large, e.g. when $\rho \in \{0.3, \ldots, 0.7\}$. In real systems it may be desirable for $\rho$ to be neither too small nor too large, since small $\rho$ results in low server utilization, and large $\rho$ in large system times. If so, PART-$n$-TSP would be good in practice as it achieves lower $\sigma[T]$ than PART-TSP and NN, and lower $E[T]$ than PART-TSP. We also prove that PART-$n$-TSP is $E[T]$ optimal under light load ($\rho \rightarrow 0^+$) and asymptotically optimal under heavy load ($\rho \rightarrow 1^-$). See Theorem 3.2.

2. Polling-Sequencing Policies. The class of Polling-Sequencing (P-S) polices include a polling phase and a sequencing phase. For the notation of P-S policies, we use “PART-” to denote the polling (partitioning) phase, followed with the sequencing policies for the sequencing phase. For example, PART-TSP means first partition the region $A$ into polling stations, and use TSP to sequence the tasks inside each polling station. Similarly, we can define PART-NN, PART-SJF, etc.

2.1. Spatial-Polling: Markov Chain. Polling policies are well-established in the queueing theory literature. Overviews and surveys of polling systems can be found in [1, 16, 21, 7]. Stability and ergodicity criteria for polling systems are well established and can be found in [1, 16, 21, 7].

The polling phase of the P-S class is a spatial-polling policy, we divide the region $A$ into an $r$-partition $\{A^k\}_{k=1}^r$, each of area $A^k$. We label the $r$ partitions as $1, 2, \ldots, r$. We regard each partition as a station in classic polling systems. In this way we generalize the polling system [1, 16] in classic queueing theory to the spatial case.

The vehicle visits the partitions in cyclic order, $1, 2, \ldots, r, 1, 2, \ldots$, and serves the tasks in each partition. Without loss of generality, we assume that the vehicle is initially partition 1. Thus, the $l$-th visit of the vehicle is partition $I(l) = (l - 1) \mod r + 1$, where $l \mod r$ means the remainder of the division of $l$ by $r$. The set of tasks waiting in partition $I(l)$ on the arrival of the $l$-th visit of the vehicle is called the $l$-th queue (the queue observed at $l$-th visit).

We denote by:

- $G^k(N)$ the number of tasks that are served in partition $k$ when the queue observed is of length $N$.
- $T_k^S(n)$ the total service time of $n$ tasks in partition $k$. The formal definition of function $T_k^S(.)$ will be given in section 2.2.
- $S^k(N)$ the duration of the service in partition $k$ when the queue observed is of length $N$.

(2.1) \[ S^k(N) = T_k^S(G^k(N)) \]

Function $G^k(.)$ characterizes the polling policy and $T_k^S(.)$ characterizes the sequencing policy.

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The tasks arrive at partition $k$ with a Poisson process of parameter $\lambda^k = \int_{A^k} f_X(x) \, dx \lambda$. The task sizes are i.i.d. with common distribution $B$ and mean $b$. Define $\rho^k = \lambda^k b$, and $\rho = \sum_{k=1}^r \rho^k$, $1 \leq k \leq r$. Let $N^k(t_1, t_2)$ denote the number of Poisson arrivals to
partition \( k, 1 \leq k \leq r \), during a (random) time interval \((t_1, t_2)\). \( N^k(t) \equiv N^k(0,t] \) is the number of Poisson arrivals in a time interval of length \( t \).

The \( l \)-th value of the polling system is described by the random variables \( N^k_l \), \( 1 \leq k \leq r, l \geq 1 \), where \( N^k_l \) represents the number of tasks in partition \( k \) at the \( l \)-th visit of the vehicle. Let \( N_l = (N^1_l, \ldots, N^r_l) \), taking values in \( \mathbb{N}^r \), where \( \mathbb{N} \) is the set of nonnegative integers.

Denote by \( S_l \), the station time, the time interval between the arrival times of the \( l \)-th visit and the \((l + 1)\)st visit of the vehicle.

\begin{equation}
S_l = S^{(l)}(N^{(l)}_l) + \Delta^{(l)}
\end{equation}

Denote by \( C_l \), the cycle time, the time interval between two successive arrivals of the vehicle to the same partition. \( C_l = S_l + \ldots + S_{l+r-1} \).

The arrival times, the service times, the switch times are mutually independent, and are independent of the past and present system states. We adopt the rigorous independence definitions from Fricker [16] with some changes for the spatial case.

Consider a queue service starting at stopping time \( \tau \) at partition \( k \) while \( N \) tasks are waiting and \( N^- \) tasks have already been served for the whole system. Let \( \mathcal{F}_\tau \) be any \( \sigma \)-field containing the history of the service process up to random time \( \tau \). \( \mathcal{F}_\tau \) is independent of the process \( N(\tau, \tau + .] \) of arrivals after \( \tau \) and of the task sizes \( \{B_{N^-+1}\}_{i>0} \) of the tasks that have not been served up to time \( \tau \). The following four assumptions hold for all \( k = 1, \ldots, r \).

\begin{itemize}
  \item \( A_1 \): \((G^k, S^k)\) is conditionally independent of \( \mathcal{F}_\tau \) given \( N \), and has the distribution of \((G^k(N), S^k(N))\) where the expressions of the random functions \((G^k(.), T^k_s(.))\) are taken independent of \( N \). i.e. The A-S policies do not depend on the past history of the service process such as the number of tasks being already served and the time spent serving them.
  \item \( A_2 \): \((G^k, S^k)\) is independent of 
  
  \((B_{N^-+G^k+1})_{i>0}, N(\tau + S^k, \tau + S^k + .])\), i.e. The selection of a task for service is independent of the required execution time and of possible future arrivals.
  \item \( A_3 \): \( G^k(0) = 0, S^k(0) = 0 \) and there exists \( N > 0 \) such that \( G^k(N) > 0 \). i.e. The vehicle leaves immediately a queue which is or becomes empty, but provides service with a positive probability once there are “enough” task(s) in the queue.
  \item \( A_4 \): \((G^k(N), S^k(N))\) is monotonic and contractive in \( N \). A function \( g(.) \) is contractive if for every \( x \geq y \), \( g(x) - g(y) \leq x - y \).
\end{itemize}

\( N_l \) evolve according to the following evolution equations:

\begin{equation}
N^k_{l+1} = \begin{cases} 
N^k_l + N^k(S_l), & \text{if } I(l) \neq k \\
N^k_l - G^k(N^k_l) + N^k(S_l), & \text{if } I(l) = k
\end{cases}
\end{equation}

where \( I(l) = (l - 1) \mod r + 1 \).

The spatial polling system has a Markovian structure as specified by the following two theorems, which is almost identical to the theorems given in [16].

**Theorem 2.1.** The sequence \( \{N_l\}_{l=0}^\infty \) is a Markov chain.

**Proof.** At the \( l \)-th polling instant \( \tau \), the server starts serving queue \( l \) (if not otherwise, otherwise he starts switching to queue \( l + 1 \)) according to policy \( G^{(l)} \) while the state of all queues is given by \( N_l \). The arrival processes after \( l \) are Poisson and are independent of \( \mathcal{F}_\tau \); the service times and the switch times involved after \( \tau \) are also independent of \( \mathcal{F}_\tau \). Because these quantities are mutually independent, it follows that given \( N_l \), the evolution of the system after \( \tau \) is independent of \( \mathcal{F}_\tau \), which ensures the Markov property of the sequence. \( \square \)
Remark 2.1.1. This Markov chain is in general not homogeneous because its transitions depend on \( l \) through \( G(l) \) and \( \Delta(l) \), and \( I(l) \) is different for each \( l \). One can check that theorem 2.1 also holds when the task arrival process is renewal. This guarantees the arrival processes after \( l \) are independent of \( F_\tau \).

Theorem 2.2. \( \{N_{lr+k}\}_{l=0}^\infty \) is a homogeneous, irreducible and aperiodic Markov chain with state space \( \{N\}_{k=1,\ldots,r} \), where \( r \) is the number of polling stations.

Proof. \( \{N_{lr+k}\}_{l=0}^\infty \) is a subsequence of the Markov chain \( \{N_l\}_{l=0}^\infty \) and is thus also a Markov chain which is homogeneous because \( I(lr+k) = k \) and \( G(lr+k) = G^k \) for \( l = 0,1,2,\ldots \).

It is irreducible because all states communicate. Indeed, \( (N^1,\ldots,N^r) \) can be reached in one step from the state \( (0,\ldots,0) \): this is realized when first no arrivals occur to all queues during the whole cycle but the last switch time \( \Delta^{r-1} \), and then the last switch time is positive and \( (N^1,\ldots,N^r) \) arrivals occur during it, all this having a positive probability because the arrival processes are Poisson. On the other hand, \( (0,\ldots,0) \) is reached in (possibly) many steps from any state \( (N^1,\ldots,N^r) \) with positive probability too: this is realized when there are no arrivals until it happens. By the same arguments, the state \( (0,\ldots,0) \) is aperiodic and so is the (irreducible) Markov chain. \( \square \)

2.2. Sequencing: Economy of Scale. Under a spatial-polling policy, the number and locations of tasks are determined in each polling station in each polling cycle, which is a static vehicle routing problem. The sequencing policies sequence the set of tasks in each polling station.

Definition 2.3. A policy for the 1-DTRP is called a Polling-Sequencing (P-S) policy if it runs a spatial-polling policy in region \( A \), and sequences the set of tasks in each polling station using some sequencing policy.

For a set of \( n \) tasks \( \{B_i,X_i\}_{i=1}^n \), each with size \( B_i \) and location \( X_i \), denote by \( T_D^P(\{X_i\}_{i=1}^n) \) and \( T_S^P(\{B_i,X_i\}_{i=1}^n) \) the travel time and service time for the \( n \) tasks \( \{B_i,X_i\}_{i=1}^n \) under sequencing policy \( P \).

\[
(2.4) \quad T_D^P(\{X_i\}_{i=1}^n) = \mathbb{E}_X \left[ \frac{1}{v} D^P(X, \{X_i\}_{i=1}^n) \right]
\]

where \( v \) is the vehicle speed, and \( D^P(X,\{X_i\}_{i=1}^n) \) is the distance travelled by the vehicle to serve the tasks \( \{B_i,X_i\}_{i=1}^n \) starting from a random point \( X \) in region \( A \) under sequencing policy \( P \).

\[
(2.5) \quad T_S^P(\{B_i,X_i\}_{i=1}^n) = T_D^P(\{X_i\}_{i=1}^n) + \sum_{i=1}^n B_i
\]

Define \( T_D^P(n) \equiv E(\{X_i\}) \left[ T_D^P(\{X_i\}_{i=1}^n) \right] \), and \( T_S^P(n) \equiv E(\{X_i\}) \left[ E(B_i) \left[ T_S^P(\{B_i,X_i\}_{i=1}^n) \right] \right] \), then

\[
(2.6) \quad T_S^P(n) = T_D^P(n) + nb
\]

\( T_S^k(n) \equiv T_S^P(n) \) and \( T_D^k(n) \equiv T_D^P(n) \) when sequencing policy \( P \) is used in partition \( k \).

Definition 2.4. A sequencing policy \( P \) is said to have economy of scale (EoS) if \( \frac{T_D^P(n)}{n} \) is nonincreasing in \( n \).

Definition 2.5. A scheduling policy is called non-location based if the distance between two consecutively executed tasks is i.i.d. A scheduling policy is called location based if the distance between two consecutively executed tasks is dependent.
Non-location based policies include FCFS, SJF, ROS and longest job first (LJF). Theorem 2.6(i) shows that non-location based policies satisfy economy of scale.

Location based policies include NN, furthest job first (FJF), TSP and the approximation algorithms for TSP such as Daganzo’s algorithm (DA) [11]. For location based policies, there are two categories. One category try to find a shorter path connecting the locations of the tasks, which we call smart. Examples include TSP, NN and the approximation algorithms of TSP such as Daganzo’s algorithm (DA) [11]. The other category tries to find a longer path connecting the locations of the tasks, which we call foolish. Examples include furthest job first (FJF). This category does not has EoS, and is not practical. Theorem 2.6(ii) below proves that the common policies in the smart category such as TSP, NN and DA have EoS. Other policies in this category can be checked by the similar analysis or through simulation. Theorem 2.6(ii) also proves that FJF does not satisfy EoS. NN and TSP are well known. In DA, one cuts a swath of approximate width, $w$, covering the region $A$. One possible pattern is shown in the left of Figure 2.1 with a swath of width $\sqrt{A/n}$. The vehicle visits the task locations by moving along the swath without backtracking.

Figure 2.1. Daganzo’s Algorithm, cited from [11].
Hamiltonian paths and $L_n$ is the minimum of $n!$ Hamiltonian paths. 
\[
\frac{E[L_{n+1}]}{n+1} = \int_0^\infty P \left( \frac{L_n}{n} > y \right) dy \leq \int_0^\infty P \left( \frac{L_n}{n} > y \right) dy = \frac{E[L_n]}{n}.
\]
Then \( \frac{T_{n+1}^D}{n+1} \leq \frac{T_n^D}{n} \). TSP satisfies EoS.

Under NN, when there are $n$ tasks, Let $L_{nN}$ be the length of the tour connecting the initial vehicle position and the locations of the $n$ tasks, then \( \frac{T_n^D}{n} = \frac{E[L_{nN}]}{n} \).

$L_{nN}^N$ is composed of $n$ segments, $L_{nN}^N = \sum_{i=1}^{n} D_{iN}$, label $i$ backwards such that $D_{iN}^N$ is the distance from the $(n - i)$-th point to the $(n - i + 1)$-th point when $i = 1, \ldots, n - 1$, and $D_n$ is the distance from the initial position of the vehicle to the 1st point. So $D_{nN}^N$ is the minimum of $i D_j$'s, where each $D_j$ is the distance between two random points in the region $\mathbf{A}$. Thus \( P \left( D_{i+1N}^N > y \right) \leq P \left( D_{nN}^N > y \right) \), this implies
\[
E \left[ D_{i+1N}^N \right] \leq E \left[ D_{iN}^N \right], \quad \text{thus} \quad \frac{E[L_{n+1}^N]}{n+1} = \frac{\sum_{i=1}^{n} E[D_{iN}^N]}{n+1} = \frac{\sum_{i=1}^{n} E[D_{i+1N}^N] + \sum_{i=1}^{n} E[D_{iN}^N]}{n+1} = \frac{\sum_{i=1}^{n} E[D_{i+1N}^N]}{n+1} \leq \frac{E[L_{nN}^N]}{n} = \frac{E[L_n]}{n}.
\]
So \( \frac{T_{n+1}^D}{n+1} \leq \frac{T_n^D}{n} \). NN satisfies EoS.

In [11], the swath was approximated to be a infinitely long strip of width $w$ neglecting corner effect as shown in the right two of Figure 2.1. The mean travel time per task when serving $n$ tasks, $T_n^D = \frac{n d_w}{w}$, where $d_w$ is the expected distance between two consecutive locations. Let $X$ denote the random distance between two consecutive points along the width of the strip, and $Y$ the distance along the side of the strip, then $E[X] = \frac{w}{2}$, $E[Y] = \frac{A}{w}w$ according to [11]. $d_w = E_{X,Y} \left( \sqrt{X^2 + Y^2} \right)$ for the Euclidean metric. $d_w \approx \frac{w}{2} + \frac{A}{2w} \psi \left( \frac{nw^2}{A} \right)$, where $\psi(x) = \frac{2}{x^2} (1 + x) \log(1 + x) - x$.

$w^* = \sqrt{\frac{2.95A}{n}}$ minimizes $d_w$. Substituting $w^*$, we see $d_w$ is decreasing with $n$. Thus \( \frac{T_n^D}{n} \) is nonincreasing in $n$.

![Mean travel time under TSP, NN and DA](image)

**Figure 2.2. Mean travel time under TSP, NN and DA.**

We show that FJF does not satisfy EoS by a counterexample. Consider a square of size $1 \times 1$ with uniformly distributed task locations. $T_1^D = E[D_1] = 0.52$, where $D_1 = \| X_1 - X_v \|$, where $X_i$ is the location of the $i$-th task and $X_v$ is the initial position of the vehicle. $X_i$ and $X_v$ are i.i.d. with pdf $f_X(x) = 1$. When there are two
tasks, the vehicle will choose the task further away, thus \( T_{Euler}^{p}(2) > 0.52 = T_{Eu}^{p}(1) \). Thus FJF does not satisfy EoS. 

**Remark 2.6.1.** Non-location based policies have trivial EoS in the sense that \( T_{Euler}^{p}(n) \) is a constant. 

Theorem 2.6(ii) is supported by the simulation results in Figure 2.2. The simulations are done in a square \( A \) of size 1 × 1. The task locations and the initial vehicle position are generated independently from a uniform distribution with pdf \( f_X(x) = 1 \). 

The length of the path connecting the vehicle and the tasks is calculated under TSP, NN and DA for different number of points \( n \).

**Theorem 2.7.** Under a sequencing policy \( P \) with economy of scale, \( \lim_{n \to \infty} \frac{T_{P}^{p}(n)}{n} = b_d \geq 0 \), and \( \exists M > 0 \), s.t. \( M \geq \frac{T_{P}^{p}(n)}{n} \geq b_d \) for all \( n \).

**Proof.** \( T_{P}^{p}(n) \geq 0 \) and \( T_{P}^{p}(n) \) is nonincreasing in \( n \) imply that \( \lim_{n \to \infty} \frac{T_{P}^{p}(n)}{n} \) exists, say \( b_d \).

Thus we have \( \lim_{n \to \infty} \frac{T_{P}^{p}(n)}{n} = b_d \) and \( M = \frac{T_{P}^{p}(1)}{1} \geq \frac{T_{P}^{p}(n)}{n} \geq b_d \geq 0 \).

**Remark 2.7.1.** \( b_d \) is a measure of how well the sequencing policy can take advantage of the task locations. Let \( L_n \) denote the length of the tour connecting \( n \) points in a square of area \( A \) under TSP. From [24] we know that \( \lim_{n \to \infty} \frac{L_n}{\sqrt{n}} = \beta_{TSP} \sqrt{A} \), where \( \beta_{TSP} \approx 0.72 \), thus \( b_d = \lim_{n \to \infty} \frac{T_{P}^{p}(n)}{n} = \lim_{n \to \infty} \frac{E[L_n]}{\sqrt{n}} = \lim_{n \to \infty} \beta_{TSP} \sqrt{\frac{A}{n}} = 0 \).

This implies that TSP does best in taking advantage of the task locations.

**2.3. Stability Condition.** Stability of DTRP is more complicated than in queueing theory because the stability of DTRP is policy dependent, whereas in queueing theory we have the policy-independent stability condition \( \rho < 1 \) for work conserving M/G/1 queues. Theorem 2.1 and 2.2 showed that \( \{N_t\}_{t=0}^{\infty} \) is a Markov chain, and \( \{N_{t+r+k}\}_{t=0}^{\infty} \) is a homogeneous, irreducible and aperiodic Markov chain. We check the ergodicity of \( \{N_{t+r+k}\}_{t=0}^{\infty} \) and the stability of the DTRP under the P-S policies in this section.

**Definition 2.8.** A polling policy characterized by \( G^k(\cdot) \) is called an unlimited-polling policy if \( G^k(N) \to \infty \), when \( N \to \infty \), \( k = 1, \ldots, r \).

One can check that the common polling policies such as the exhaustive and gated policies in [25] are unlimited-polling policies.

**Lemma 2.9.** (Lemma 3.1 in [1]) If for all \( 1 \leq k \leq r \) the Markov chains \( \{N_{t+r+k}\}_{t=0}^{\infty} \) are ergodic, then for all \( 1 \leq k \leq r \) \( \{N_{t+r+k}\}_{t=0}^{\infty} \) together with the sequence of station times \( \{S_{t+r+k}\}_{t=0}^{\infty} \) and the cycle times \( \{C_{t+r+k}\}_{t=0}^{\infty} \) converge weakly to finite random variables.

**Definition 2.10.** The DTRP under a P-S policy is said to be stable if all the \( r \) Markov chains \( \{N_{t+r+k}\}_{t=0}^{\infty} \) are ergodic.

**Lemma 2.11.** (Foster’s Criterion [3, p.19]): Suppose a Markov chain is irreducible and let \( E_0 \) be a finite subset of the state space \( E \). Then the chain is positive recurrent if for some \( h : E \to \mathbb{R} \) and some \( \epsilon > 0 \) we have \( \inf_{x} h(x) > -\infty \) and

\[ \sum_{j \in E_0} p_{jk} h(k) < \infty, j \in E_0, \]
\[ \sum_{j \in E_0} p_{jk} h(k) \leq h(j) - \epsilon, j \notin E_0, \]

where \( p_{jk} \) is the transition probability of the chain.

**Theorem 2.12.** (Stability theorem): For any P-S policy with polling policy satisfying Definition 2.8 (unlimited-polling) and sequencing policy \( P \) satisfying \( \lim_{n \to \infty} \frac{T_{P}^{p}(n)}{n} = b_d \) in each partition \( A^k \), assuming the partitions \( \{A^k\}_{k=1}^{R} \) are divided such that \( b_d = \lim_{n \to \infty} \frac{T_{P}^{p}(k)}{n} = b_d, \forall 1 \leq k \leq r \), then when \( \rho + \lambda b_d < 1 \), the Markov chains
\( \{N_{r+k}\}_{k=0}^{\infty} \) are ergodic, \( \forall 1 \leq k \leq r \). Moreover, if the sequencing policy \( P \) satisfies Definition 2.4 (EoS), then \( \rho + \lambda b_0 < 1 \) is necessary for the ergodicity of \( \{N_{r+k}\}_{k=0}^{\infty} \).

Proof. Sufficieny: taking a conditional expectation in (2.3), summing over \( k \), and substituting (2.2) and (2.1) we obtain:

\[
E \left[ \sum_{k=1}^{r} b N_{k+1} | N_i \right] = \sum_{k=1}^{r} b N_{k}^r - b G^{I(I)} \left( N_{I(I)}^r \right) + E \left[ \sum_{k=1}^{r} b N_{k} \left( \Delta_{I(I)} \right) | N_i \right] + E \left[ \sum_{k=1}^{r} b N_{k} \left( T_{S}^{I(I)} \left( G^{I(I)} \left( N_{I(I)}^r \right) \right) \right) | N_i \right]
\]

Define \( \gamma^k = \rho - 1 + \lambda T_{E}^{I(I)} \left( \frac{G^{I(I)}(N_{I(I)}^{I(I)+k})}{N_{I(I)}^{I(I)+k}} \right) \), \( k = 0, \ldots, r - 1 \), then

\[
E \left[ \sum_{k=1}^{r} b N_{k+1} | N_i \right] = \sum_{k=1}^{r} b N_{k}^r + \rho \delta(I(I)) + \gamma b G^{I(I)} \left( N_{I(I)}^r \right).
\]

Similarly,

\[
E \left[ \sum_{k=1}^{r} b N_{k+1} | N_i \right] = E \left[ E \left[ \sum_{k=1}^{r} b N_{k+2} | N_{I+1}, N_i \right] | N_i \right] = E \left[ \sum_{k=1}^{r} b N_{k+2} | N_{I+1} \right] + \rho \delta(I(I+1)) + E \left[ \gamma b G^{I(I+1)} \left( N_{I(I+1)}^r \right) | N_i \right].
\]

Since \( N_{I(I+1)}^r = N_{I(I+1)}^r + N_{I(I+1)}^r (S_i) \geq N_{I(I+1)}^r \), and \( G_k(\cdot) \) is nondecreasing, then

\[
E \left[ G^{I(I)} \left( N_{I(I+1)}^r \right) | N_i \right] \geq E \left[ G^{I(I)} \left( N_{I(I+1)}^r \right) | N_i \right] = G^{I(I)} \left( N_{I(I+1)}^r \right),
\]

\[
\rho + \lambda b < 1 \implies \epsilon_1 = \frac{1 - \rho - \lambda b}{\lambda} > 0.
\]

Since \( \lim_{n \to \infty} \frac{T_{E}^{I(I)}(n)}{n} = b_1 \geq 0 \),

then \( \exists M > 0 \), s.t. \( n > M \) implies \( \frac{T_{E}^{I(I)}(n)}{n} - b_1 < \epsilon_1 \), i.e. \( \rho - 1 + \lambda T_{E}^{I(I)}(n) < 0 \).

Thus when \( G^{I(I)}(N_{I(I+1)}^r) > M_1, G^{I(I)}(N_{I(I+1)}^r) > M_1, \gamma^i < 0 \).

This implies \( E \left[ \gamma b G^{I(I+1)} \left( N_{I(I+1)}^r \right) | N_i \right] \leq \gamma b G^{I(I+1)} \left( N_{I(I+1)}^r \right) \).

So when \( G^{I(I)} \left( N_{I(I+1)}^r \right) > M_1, \)

\[
E \left[ \sum_{k=1}^{r} b N_{k+2} | N_i \right] \leq E \left[ \sum_{k=1}^{r} b N_{k}^r | N_i \right] + \rho \delta(I(I)) + \gamma b G^{I(I+1)} \left( N_{I(I+1)}^r \right) = \sum_{k=1}^{r} b N_{k}^r + \rho \left( \delta(I(I)) + \delta(I(I+1)) + \gamma b G^{I(I+1)} \left( N_{I(I+1)}^r \right) + \gamma b G^{I(I+1)} \left( N_{I(I+1)}^r \right) \right).
\]

Repeating the above calculation, we obtain

\[
E \left[ \sum_{k=1}^{r} b N_{k+1} | N_i \right] \leq \sum_{k=1}^{r} b N_{k}^r + \rho \delta + \sum_{k=0}^{r-1} \gamma b G^{I(I+k)} \left( N_{I(I+k)}^r \right),
\]

when \( G^{I(I+k)} \left( N_{I(I+k)}^r \right) > M_1, k = 1, \ldots, r - 1 \).

Since \( \gamma^k < 0 \), when \( G^{I(I+k)} \left( N_{I(I+k)}^r \right) > M_1, k = 0, \ldots, r - 1 \),

then \( \exists M > M_1, \) s.t.

\[
G^{I(I+k)} \left( N_{I(I+k)}^r \right) > M \implies \epsilon = \rho \delta + \sum_{k=0}^{r-1} \gamma b G^{I(I+k)} \left( N_{I(I+k)}^r \right) < 0.
\]
Define \( E_0 = \{ N_l \in \mathbb{N}^r \mid G_l^{(l+i)}(N_l^{(l+i)}) \leq M, k = 1, \ldots, r \} \), then \( E_0 \) is a finite subset of the state space \( \mathbb{N}^r \).

Define \( h(N) = \sum_{k=1}^r bN_k \), since \( b \geq 0 \) and \( N \in \mathbb{N}^r \), then \( \inf_N h(N) = -\infty \).

It then follows that
\[
E[h(N_t + r) \mid N_l] \leq \sum_{k=1}^r bN_k^l + \rho \delta + \sum_{k=0}^{r-1} \gamma_k bG_l^{(l+i)}(N_k^{(l+i)})
\]
when \( N_l \in E_0 \).

Then \( \{ N_{l+r} \}_{r=0}^\infty \) is positive recurrent by Lemma 2.11 (Foster’s Criterion), thus it is ergodic (irreducible, aperiodic and positive recurrent).

Necessity when economy of scale applies: Bertsimas et al. gave the necessary condition for stability in [5] \( \rho + \lambda \frac{\overline{d}}{n} \leq 1 \), where \( \overline{d} = \lim_{n \to \infty} E[D_i] \), where \( D_i \) denotes the distance traveled from task \( i \) to the next task served after \( i \), i.e. \( \overline{d} \) is the steady state expected value of \( D_i \). Let \( N^k \) be the number of tasks served in partition \( k \) in steady state. \( P(N^k = n, X_i \in A^k) \) denotes the probability that there are \( n \) tasks served in partition \( k \) in steady state and task \( i \) is one of them. Then
\[
\frac{d}{n} = \sum_{k=1}^r \sum_{n=1}^\infty \frac{P_k^{\infty}(n) + \Delta}{n} P(N^k = n, X_i \in A^k)
\]
\[
> \sum_{k=1}^r \sum_{n=1}^\infty \lim_{n \to \infty} \frac{P_k^{\infty}(n)}{n} P(N^k = n, X_i \in A^k)
\]
\[
= b_d \sum_{k=1}^r \sum_{n=1}^\infty P(N^k = n, X_i \in A^k) = b_d.
\]

So \( \rho + \lambda b_d \leq \rho + \lambda \frac{\overline{d}}{n} < 1 \). \( \Box \)

Remark 2.12.1. The stability condition \( \rho + \lambda b_d < 1 \) has an additional term \( \lambda b_d \) compared to \( \rho < 1 \) in queueing theory, where \( b_d \) is the mean travel time per task when \( n \to \infty \).

Remark 2.12.2. By Lemma 2.9, ergodicity implies that the sequence of station times \( \{ S_{l+r+k} \}_{r=0}^\infty \) and the cycle times \( \{ C_{l+r+k} \}_{r=0}^\infty \) converge weakly to finite random variables. The i-th task arriving in partition \( k \) to be served in station time \( S_{l+r+k} \) first spends time \( W_{Oi} \) to wait outside the previous cycle, \( C_{(i-1)r+k} \), and spends time \( W_{Ii} \) inside the current station time \( S_{l+r+k} \). \( W_{Oi} \) and \( W_{Ii} \) are well defined based on \( C_{(i-1)r+k} \) and \( S_{l+r+k} \) under the P-S policy, and \( W_{Oi} \leq C_{(i-1)r+k} \) and \( W_{Ii} \leq S_{l+r+k} \). So \( W_{Oi} \) and \( W_{Ii} \) converge weakly to finite random variables. Thus the system time \( T_i = W_{Oi} + W_{Ii} \) converges weakly to finite random variable.

3. System Time Distribution. In the last section, we give a necessary and sufficient condition for the stability of the Dynamic Traveling Repairman Problem (DTRP) for the class of Polling-Sequencing (P-S) policies satisfying unlimited-polling and economy of scale. When the DTRP is stable, the distribution of the steady state system time \( T \) exists.

3.1. PART-n-Traveling Salesman Policy. Bertsimas et al. [4] introduced the traveling salesman policy (TSP). It is based on collecting tasks into sets of size \( n \) that are then served in a TSP path. To be precise, we call this the \( n \)-TSP. This policy is a one-partition policy in the P-S class. We generalize it to multiple partitions as follows and call it the PART-\( n \)-TSP.

Definition 3.1. A Polling-Sequencing policy in Definition 2.3 is call the PART-\( n \)-TSP policy if the sequencing phase is an \( n \)-TSP policy that collects tasks into sets of cardinality \( n \), and then serve them using an optimal traveling salesman path.

The polling phase involving generating an \( r \)-partition \( \{ A^k \}_{k=1}^r \) of \( A \) that is simultaneously equitable with respect to \( f(x) \). In particular, when the region \( A \) is a square region \( A \) with size \( a \times a \), and the tasks are uniformly distributed in \( A \) with pdf \( f(x) = \frac{1}{a} \), \( A \) is divided into \( r = m^2 \) square partitions, each has size \( \frac{a}{m} \times \frac{a}{m} \), where \( m > 1 \) is a given integer that parameterizes the policy. The vehicle visits the
partitions in a cyclic order. Partitions are numbered so that for any \( k = 1, \ldots, r - 1 \), partition \( k + 1 \) is adjacent to partition \( k \), and partition \( r \) is adjacent to partition 1 when \( m \) is even, or to the diagonal of partition 1 when \( m \) is odd, as illustrated in Figure 3.1 for the case \( m = 4 \) and \( m = 5 \). The vehicle cycles through the partitions in the order 1, \ldots, r, 1, \ldots, r, \ldots. After the vehicle finishes the tasks polled in the current partition under a sequencing policy, the vehicle moves to an adjacent partition and serves it under the same sequencing policy.

To move from one partition (polling station) to the next, the vehicle uses the projection rule shown in Figure 3.2 as introduced in [4]. Its last location in a given partition is simply “projected” onto the next partition to determine the server’s new starting location. The vehicle then travels in a straight line between these two locations. This makes the distance traveled between partitions a constant, each starting location uniformly distributed, and independent of the locations of tasks in the new partition. In practice, one might use a more intelligent rule such as moving directly to the first task in the next partition. When \( m \) is even, the vehicle always travels to an adjacent partition. Thus the switch time between two consecutive partitions is always \( \Delta^k = \frac{a}{m} \), \( k = 1, \ldots, r \). Thus \( \Delta = \sum_{k=1}^{r} \Delta^k = ma \). When \( m \) is odd, the vehicle always travels to an adjacent partition except the last one. Thus \( \Delta^k = \frac{a}{m} \) when \( k = 1, \ldots, r - 1 \), and \( \Delta^r = \frac{\sqrt{2}a}{m} \) as shown in Figure 3.1. Thus \( \Delta = \sum_{k=1}^{r} \Delta^k = \frac{(m^2 - 1 + \sqrt{2})a}{m} \). Then we have
\[ \Delta = \begin{cases} \frac{ma}{m} & \text{if } m \text{ is even}, \\
\frac{(n^2-1+\sqrt{7})a}{m} & \text{if } m \text{ is odd}. \end{cases} \]

In the sequencing phase, we use the exhaustive \(n\)-TSP policy to sequence the tasks in each partition. This policy is adopted from the TSP policy in [4]. We repeat it for the convenience of the reader. Let \(N_l^k\) denote the \(l\)-th set of \(n\) tasks to arrive in partition \(k\). Each set \(N_l^k\) has cardinality \(n\). For example, \(N_1^1\) is the set of tasks \(1, \ldots, n\) in partition \(k\), and \(N_2^k\) is the set of tasks \(n+1, \ldots 2n\) in partition \(k\), and so on. To serve a set, we form a TSP path of the \(n\) tasks in the set starting at the initial position of the vehicle and ending at the location of the last task in the TSP path. A TSP path is the Hamiltonian path with the minimum length among all the Hamiltonian paths. A Hamiltonian path is a path that visits each task location exactly once starting at the initial position of the vehicle.

The vehicle starts at some location in partition 1. If all tasks in set \(N_1^1\) have arrived, we form a TSP path over these tasks. Tasks are then served by following the TSP path. A TSP path is the Hamiltonian path with the minimum length among all the Hamiltonian paths. A Hamiltonian path is a path that visits each task location exactly once starting at the initial position of the vehicle.

The sets \(N_l^k\) are also served using a TSP path. Otherwise, the vehicle moves to partition 2, and so on. To serve a set, we form a TSP path of the \(n\) tasks in the set starting at the initial position of the vehicle and ending at the location of the last task in the TSP path. A TSP path is the Hamiltonian path with the minimum length among all the Hamiltonian paths. A Hamiltonian path is a path that visits each task location exactly once starting at the initial position of the vehicle.

We know \(\rho + \lambda b_d < 1\) is the stability condition by Theorem 2.12, and the sequencing phase TSP has \(b_d = 0\) by Remark 2.7.1. Thus PART-\(n\)-TSP is stable if and only if \(\rho < 1\).

### 3.2. Calculation of System Time Distribution

The system time \(T\) of a task has three components:

- \(W_O\), the time a task waits for its set to form (wait for the last task in the set to arrive).
- \(W_P\), waiting time of the set in the polling system.
- \(W_I\), the time it takes to complete service of the task once the task’s set enters service.

Thus,

\[ T = W_O + W_P + W_I \]

where \(W_O\), \(W_P\) and \(W_I\) are independent. The distribution of \(T\) can be obtained from the distributions of \(W_O\), \(W_P\) and \(W_I\) through convolution.

#### 3.2.1. Distribution of \(W_O\)

We first obtain the distribution of \(W_O\), together with its expectation and variance. Pick a random task. Let \(W_{OI}\) be the waiting time of a task outside a set if it is the \((n-l)\)-th task arrived in the set, \(l = 0, \ldots, n-1\). Since we have equitable partitions, then the task arrival process inside each partition is Poisson with arrival rate \(\frac{\lambda}{r}\). Thus \(W_{OO} = 0\) and \(W_{OI}\) is Erlang distributed with parameters \((l, \frac{\lambda}{r})\), \(l = 1, \ldots, n-1\). Thus the cdf of \(W_{OI}\), \(F_{W_{OI}}(t; l, \frac{\lambda}{r}) = 1 - \sum_{j=0}^{l-1} \frac{1}{j!} e^{-\frac{\lambda}{r}t} \left(\frac{\lambda}{r}t\right)^j\),

\[ E[W_{OI}^2] = \frac{(l^2+l)r^2}{\lambda^2}, \text{ and } E[W_{OI}^2] = \frac{(l^2+l)r^2}{\lambda^2}. \]

Since it is equally probable that a task is the \((n-l)\)-th arrived task in the set, then

\[ P(W_O \leq t) = \sum_{l=0}^{n-1} P(W_{OI} \leq t) \frac{1}{n} = \frac{1}{n} \left(1 + \sum_{l=1}^{n-1} \left(1 - \sum_{j=0}^{l-1} \frac{1}{j!} e^{-\frac{\lambda}{r}t} \left(\frac{\lambda}{r}t\right)^j\right)\right). \]

\[ E[W_O] = \sum_{l=0}^{n-1} E[W_{OI}] \frac{1}{n} = \frac{1}{n} \sum_{l=0}^{n-1} \frac{lr}{\lambda} = \frac{(n-1)r}{2\lambda}. \]
\[
E[W_O^2] = \sum_{l=0}^{n-1} E[W_{O_l}^2] \frac{1}{n} = \frac{1}{n} \sum_{l=0}^{n-1} \left( \frac{l^2 + t}{\lambda^2} \right) = \frac{(n^2 - 1)r^2}{3\lambda^2}.
\]
\[
Var[W_O] = E[W_O^2] - E[W_O]^2 = \frac{(n^2 + 6n - 7)r^2}{12\lambda^2}.
\]

To sum up,

\[
P(W_O \leq t) = \frac{1}{n} \left( 1 + \sum_{l=1}^{n-1} \left( 1 - \sum_{j=0}^{l-1} \frac{1}{j!} e^{-\frac{\lambda}{r} t} \left( \frac{\lambda}{r} t \right)^j \right) \right)
\]

(3.3)

\[
E[W_O] = \frac{(n - 1)r}{2\lambda}
\]

(3.4)

\[
Var[W_O] = \frac{(n^2 + 6n - 7)r^2}{12\lambda^2}
\]

(3.5)

### 3.2.2. Distribution of \(W_I\)

Before discussing \(W_P\), we compute the distribution of \(W_I\), together with \(E[W_I]\) and \(Var[W_I]\) as follows. Let \(W_{Inj}\) be the waiting time of a task inside a set if it is the \(j\)-th task to be served in the set of \(n\) tasks, \(j = 1, \ldots, n\). Then

\[
W_{Inj} = D_{nj} + \sum_{i=1}^{j} B_i
\]

(3.6)

where \(D_{nj}\) is the travel distance from the initial vehicle position to the location of the \(j\)-th task through a TSP path in a partition. \(B_i\) is i.i.d.. \(D_{nj}\) is independent of \(B_i\). Thus \(\sum_{j=1}^{k} B_i\) is the convolution of \(j\) \(B_i\)'s, and \(W_{Inj}\) is the convolution of \(\frac{D_{nj}}{r}\) and \(\sum_{i=1}^{k} B_i\).

Let \(L_{nj}\) be the \(D_{nj}\) value when there is only one partition, i.e., \(r = 1\). When \(r > 1\), we assume that

\[
D_{nj} = d \frac{cL_{nj}}{\sqrt{r}}
\]

(3.7)

where \(= d\) means identically distributed with, and \(c\) is a positive constant. In particular, when region \(A\) and all the partitions \(A^k\) are squares, and \(r = m^2\) with \(m\) an integer, \(c = 1\).

We obtain the empirical distribution of \(L_{nj}\), together with \(E[L_{nj}]\) and \(Var[L_{nj}]\) for different \(n\) and \(j = 1, \ldots, n\) through simulation on the TSP path. The ant colony optimization algorithm [14] is used to heuristically search for the TSP path. Both the number of ants and the number of iterations are set to 1000. The distribution of \(D_{nj}\) is calculated from \(L_{nj}\) by scaling in (3.7). We write \(L_n\) for \(L_{nn}\) and \(D_n\) for \(D_{nn}\).

Figure 3.3 shows the values of \(E[L_n]\) and \(Var[L_n]\) for different \(n\) when the region is a square of size 1 \(\times\) 1.

Figure 3.4 shows the pdf of \(L_{nj}\) for a set of \(n = 5\) tasks. The tasks are uniformly distributed on a square of 1 \(\times\) 1.

Since it is equally probable that the task is the \(k\)-th served task in the TSP path, \(k = 1, \ldots, n\), then

\[
P(W_I \leq t) = \frac{1}{n} \sum_{j=1}^{n} P(W_{Inj} \leq t)
\]

(3.8)
Figure 3.3. Expectation and variance of the length of the TSP path for \( n \) tasks.

Figure 3.4. The pdf of \( W_{lnj} \) and \( W_I \).

Then the pdf of \( W_I \), \( f_{W_I}(t) = \frac{1}{n} \sum_{j=1}^{n} f_{W_{lnj}}(t) \).

\[
E[W_I] = \frac{1}{n} \sum_{j=1}^{n} E[W_{lnj}],
\]

\[
E[W_I^2] = \frac{1}{n} \sum_{j=1}^{n} E[W_{lnj}^2],
\]

\[
\]

The distribution of \( W_I \) can be calculated from (3.6) and (3.8). \( W_I \) does not have a closed form, but can be arbitrarily accurate through simulation. Observe that in order to obtain the distribution of \( D_{nj} \) and \( W_I \) for partitions with different sizes parameterized by \( r \), we do not need to rerun the simulation for each partition with
different size. We only need to run it once for $L_{nj}$ and store the data. $D_{nj}$ and $W_I$ are obtained by scaling and convolution.

### 3.2.3. Distribution of $W_P$.

The analysis of $W_P$ uses the results from [15] by establishing the PART-$n$-TSP to be equivalent to a classic polling system over jobs that are the sets $N_i^k$.

Since the task arrival process is Poisson with arrival rate $\lambda$, the distribution of the interarrival time of sets, $A$, is Erlang of order $n$ and arrival rate $\lambda$, i.e., $A \sim \text{Erlang}(n, \lambda)$. Let $A_k$ be the interarrival time of sets that fall in partition $A^k$. Then $A_k \sim \text{Erlang}(n, \frac{\lambda}{r})$. Thus

$$E[A_k] = \frac{nr}{\lambda}, \text{Var}[A_k] = \frac{nr^2}{\lambda^2} \tag{3.9}$$

The arrival rate of a set is

$$\lambda^s = \frac{\lambda}{n} \tag{3.10}$$

The arrival rate of a set in partition $A^k$, $k = 1, \ldots, r$, is

$$\lambda_k^s = \frac{\lambda^s}{r} = \frac{\lambda}{nr} \tag{3.11}$$

The size of a set, or the time needed to travel to and execute all the tasks in the set, is $W_{lnn}$ as given in (3.6). We write $W_n$ for $W_{lnn}$. The size of each set $W_n$ is i.i.d.. Thus, if we treat each set as a job with size $W_n$, and each partition as a polling station, then the system is a classic polling system on $r$ polling stations with renewal (Erlang) arrival of rate $\lambda^s$, job size $W_n$, and switch time $\Delta^k$. The load is

$$\rho^s = \lambda^s E[W_n] \tag{3.12}$$

The load in partition $A^k$, $k = 1, \ldots, r$, is

$$\rho_k^s = \frac{\rho^s}{r} \tag{3.13}$$

$W_P$ is the waiting time of each set (job) in this classic polling system. Exhaustive or gated PART-$n$-TSP correspond to exhaustive or gated FCFS on sets, respectively.

Dorsman et al. [15] provide closed form approximations for the distribution of the steady state waiting time of a job, $W_P$, for polling systems under a renewal arrival process with gated or exhaustive policies when the sequencing policy is FCFS. They claim that for exhaustive-FCFS policies,

$$P(W_P \leq t) \approx P(UI \leq (1 - \rho^s)t) \tag{3.14}$$

where $U$ is uniformly distributed on $[0, 1]$, and $I$ is Gamma distributed with parameters

$$\alpha = \frac{2E[\Delta]\delta}{\sigma^2} + 1, \beta = \frac{2E[\Delta]\delta + \sigma^2}{2\sigma^2(1 - \rho^s)E[WB_{Boon}]} \tag{3.15}$$

where $\Delta = \sum_{k=1}^r \Delta^k$ is the total switch time in a cycle. When the region $A$ and partitions $A^k$ are squares as shown in Figure 3.1, $\Delta$ is given in (3.1). $\rho^s$ is given in (3.12).
To explain $\delta$, $\sigma^2$ and $E[W_{Boon}]$, we denote by $\hat{y}$ the value of each variable $y$ that is a function of $\rho^s$ evaluated at $\rho^s = 1$. $\delta = \sum_{j=1}^{r} \sum_{k=j+1}^{r} \hat{\rho}^s_j \hat{\rho}^s_k$, where $\hat{\rho}^s_k$ is given in (3.13) evaluated at $\rho^s = 1$. Since we have equitable partitions, then

$$\hat{\rho}^s_k = \frac{1}{r} \tag{3.16}$$

for all $k = 1, \ldots, r$. Thus

$$\delta = \frac{r(r-1)}{2r^2} = \frac{r-1}{2r}. \tag{3.17}$$

Again by [15]

$$\sigma^2 = \sum_{k=1}^{r} \hat{\lambda}^s_k \left( Var [W_n] + \hat{\rho}^s_k Var [\hat{A}_k] \right). \tag{3.18}$$

Since $\hat{\lambda} = n \hat{\lambda}^s$ by (3.11), then $Var [\hat{A}_k] = \frac{\nu^2}{\lambda^2} = \frac{\nu^2}{n\lambda^2}$ by (3.9). Also, $\hat{\lambda}^s_k = \frac{\hat{\lambda}}{r}$ by (3.11), then substituting (3.16) we have

$$\sigma^2 = \hat{\lambda}^s \left( Var [W_n] + \frac{1}{n\lambda^2} \right), \tag{3.19}$$

where by (3.12)

$$\hat{\lambda}^s = \frac{1}{E[W_n]} \tag{3.20}$$

From (3.6) we know

$$E[W_n] = \frac{E[D_n]}{v} + nb \tag{3.21}$$

$$Var [W_n] = \frac{Var[D_n]}{v^2} + n\sigma_B^2$$

where $E[D_n]$ and $Var[D_n]$ are obtained from $E[L_n]$ and $Var[L_n]$ by (3.7), and $E[L_n]$ and $Var[L_n]$ are obtained from simulation as shown in Figure 3.3. Thus $\sigma^2$ is known substituting (3.19), (3.20) and (3.21).

Finally by Boon et al. [6], for equitable partitions

$$E[W_{Boon}] = \frac{K_0 + K_1 \rho^s + K_2 (\rho^s)^2}{1 - \rho^s} \tag{3.22}$$

where $K_0 = E[\Delta^+]$. $\Delta^+$ is called the residual of the random variable $\Delta$ with $E[\Delta^+] = E[\Delta^2]/2E[\Delta]$. In our case, $\Delta$ is deterministic. Thus,

$$K_0 = E[\Delta^+] = \frac{\Delta}{2} \tag{3.23}$$

with $\Delta$ given in (3.1). $K_1 = \hat{\rho}^s_k \left( \left( \frac{\hat{c}^2_{A_k}}{\hat{c}^2_{A_k}} \right)^4 I\{c^2_{A_k} \leq 1\} + 2 \frac{c^3_{A_k}}{\hat{c}^2_{A_k}} I\{c^3_{A_k} > 1\} - 1 \right) E[W_n^+] + E[W_n^+] + \hat{\rho}^s_k (E[\Delta^+] - E[\Delta])$, where

$$\hat{c}^2_{A_k} = \frac{Var[\hat{A}_k]}{E[\hat{A}_k]^2} \tag{3.24}$$
and $\mathbb{1}\{\Omega\}$ is the indicator function defined as $\mathbb{1}\{\Omega\} = 1$ if $\Omega$ is true, and $= 0$ otherwise. By (3.9) we have

$$c^2_{A_k} = \frac{1}{n}.$$  

(3.25)

Substituting (3.1), (3.16), (3.23) and (3.25) we have

$$K_1 = E[W^+_n] \left( \frac{1}{rn^4} - \frac{1}{r} + 1 \right) - \frac{\Delta}{2r},$$  

(3.26)

where, by definition of a residual,

$$E[W^+_n] = \frac{E[W^2_n]}{2E[W_n]} = \frac{Var[W_n] + E[W_n]^2}{2E[W_n]}$$  

(3.27)

with $E[W_n]$ and $Var[W_n]$ given in (3.20) and (3.21). Finally, by [6]

$$K_2 = 1 - \hat{\rho}^s_k \left( \frac{\sigma^2}{25} + E[\Delta] \right) - K_0 - K_1$$  

(3.28)

is known by substituting (3.1), (3.16), (3.17), (3.18), (3.23) and (3.26). Thus, we can calculate the closed form approximation for the cdf of $W_P$ by (3.14).

After obtaining the distributions of $W_O$, $W_P$ and $W_I$, we are able to calculate the distribution of $T = W_O + W_P + W_I$ through convolution. In the three components of $T$, $W_O$ is accurate and known in closed form in (3.3). $W_P$ has a closed form approximation in (3.14). $W_I$ does not have a closed form, but can be arbitrarily accurate through simulation of TSP paths. It is also easy to obtain $W_I$ because we only need to run the simulation once for the empirical distribution of $L_{nj}$ and store the data, then calculate the distribution of $D_{nj}$ and $W_I$ by scaling in (3.7) and convolution in (3.8) for partitions with different sizes parameterized by $r$.

Figure 3.5 shows the pdf of $T$ obtained from the convolution of its three components, and the empirical pdf of $T$ obtained through simulation under the exhaustive PART-$n$-TSP when the region $A$ is a square of size $1 \times 1$ with $m = 2$ and $n = 5$, and $m = 2$ and $n = 10$, separately. The tasks are uniformly distributed in the square with size $B_i \sim Unif[0, 0.5]$. The tasks arrive according to a Poisson process with rate $\lambda = 1$. Thus load $\rho = \lambda b = 0.5$. The simulated empirical pdf of $T$ is regarded as the “true” value. We can see that the approximated values are very close to the true (simulated) values.

### 3.3. Comparison of PART-$n$-TSP, PART-TSP and Nearest Neighbor.

Bertsimas et al. [4] compared the $E[T]$ of SQM, FCFS, PART-FCFS, SFC, NN and $n$-TSP through simulation, and concluded that NN achieves lower $E[T]$ than other policies simulated. PART-TSP [31] or DC [20] were proven to be $E[T]$ optimal under the light and heavy loads. Here the focus is on $Var[T]$ or $\sigma[T]$.

Since the approximation for the cdf of $T$ for PART-$n$-TSP is both good and easy to compute, we can optimize the two parameters $n$ and $r$ to minimize $E[T]$ or other performance metrics when the region $A$ and partitions $A^k$ are squares. Table 3.1 gives the $r^*$ and $n^*$ in the range $r \in \{1^2, 2^2, \ldots, 10^2\}$ and $n = \{1, \ldots, 60\}$ that minimize $E[T]$ under exhaustive PART-$n$-TSP and the corresponding $E[T]$ and $\sigma[T]$ values for different $\rho$ values. The region is a square of size $1 \times 1$ and $B_i \sim Unif[0, 0.5]$. Noting that FCFS is PART-$n$-TSP when $r = 1$ and $n = 1$, PART-FCFS is PART-$n$-TSP.
when \( n = 1 \), and \( n-TSP \) is \( PART-n-TSP \) when \( r = 1 \). Thus by optimizing on \( r \) and \( n \), \( PART-n-TSP \) has better performance than FCFS, PART-FCFS and \( n-TSP \).

We compare \( PART-n-TSP \) with PART-TSP [31] and Nearest Neighbor [4] since they are considered near optimal in the literature. We simulate PART-TSP and Nearest Neighbor under the same setting. The number of partitions for PART-TSP is set to be the optimal number of partitions for \( PART-n-TSP \). The average number of tasks served inside each gate, denoted by \( E[n] \), is also shown in the table. We generate \( N = 100,000 \) tasks are for each load \( \rho \) value. Only the 25,000th to the 75,000th tasks are used to calculate \( E[T] \) and \( \sigma[T] \) to make sure that the steady state data are used. We have checked this by randomly sampling time segments in this range. Figure 3.6 shows the truncation of 1000 data points for PART-TSP when \( \rho = 0.9 \), \( B_i \sim [0, 0.5] \), and \( r = 25 \). Each time segments of about 50 data is the number of tasks inside each gate, indeed this is true as shown in Table 3.1 \( E[n] = 49.7 \) for this case. The system time is decreasing in general in each segment because tasks arrive earlier in a gate wait more than those arrive later. Table 3.2 gives the case when \( B_i \sim Uni[0, 1] \).

The \( E[T] \) and \( \sigma[T] \) of PART-TSP and NN are compared to those of \( PART-n-TSP \). The percentage following the \( E[T] \) and \( \sigma[T] \) of PART-TSP and NN are the ratio of these values over those of \( PART-n-TSP \). The minimum \( E[T] \) and \( \sigma[T] \) at each load level of the three policies are bolded. From Tables 3.1 and 3.2, we can see that NN achieves lower \( E[T] \) than \( PART-n-TSP \) and PART-TSP for all \( \rho \in \{0.1 \ldots 0.9\} \) in both \( B_i \sim [0, 0.5] \) and \( B_i \sim [0, 1] \). \( PART-n-TSP \) achieves lower \( E[T] \) than PART-TSP when \( \rho \) is not too small or too large, e.g. when \( \rho \in \{0.3, \ldots, 0.7\} \) for \( B_i \sim [0, 0.5] \), and when \( \rho \in \{0.4, \ldots, 0.8\} \) for \( B_i \sim [0, 1] \). \( PART-n-TSP \) has higher \( E[T] \) than PART-TSP when \( \rho \) is low because it is better to have the average number of tasks in a set to be between 1 and 2 as done by PART-TSP, but \( PART-n-TSP \) can only set it to be either 1 or 2, resulting in higher \( E[T] \). \( PART-n-TSP \) has higher \( E[T] \) than PART-TSP when \( \rho \) is high because \( r^* > 1 \) when \( \rho \) is high. Then there is a switching time between partitions. By setting \( n \) to be a fixed number under \( PART-n-TSP \), the vehicle might
Table 3.1
Comparison of PART-n-TSP, PART-TSP and Nearest Neighbor on $E[T]$ and $\sigma[T]$: $B_i \sim \text{Unif}[0, 0.5]$

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r^*$ for PART-n-TSP</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>25</td>
</tr>
<tr>
<td>$n^*$ for PART-n-TSP</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>12</td>
<td>24</td>
<td>57</td>
<td>59</td>
<td>57</td>
</tr>
<tr>
<td>$E[n]$ for PART-TSP</td>
<td>1.05</td>
<td>1.25</td>
<td>1.75</td>
<td>3.46</td>
<td>9.08</td>
<td>23.3</td>
<td>61.2</td>
<td>63.9</td>
<td>49.7</td>
</tr>
<tr>
<td>$E[T]$ for PART-n-TSP</td>
<td>1.04</td>
<td>1.71</td>
<td>1.70</td>
<td>2.80</td>
<td>5.90</td>
<td>9.99</td>
<td>20.3</td>
<td>81.4</td>
<td>321</td>
</tr>
<tr>
<td>$E[T]$ for PART-TSP</td>
<td>0.95 (91%)</td>
<td>1.26 (74%)</td>
<td>1.87 (110%)</td>
<td>3.30 (118%)</td>
<td>5.97 (101%)</td>
<td>10.9 (109%)</td>
<td>24.4 (120%)</td>
<td>51.7 (64%)</td>
<td>181 (56%)</td>
</tr>
<tr>
<td>$E[T]$ for NN</td>
<td>0.94 (90%)</td>
<td>1.21 (71%)</td>
<td>1.66 (98%)</td>
<td>2.46 (88%)</td>
<td>3.81 (65%)</td>
<td>6.37 (64%)</td>
<td>12.7 (63%)</td>
<td>32.6 (40%)</td>
<td>154 (48%)</td>
</tr>
<tr>
<td>$\sigma[T]$ for PART-n-TSP</td>
<td>0.69</td>
<td>1.19</td>
<td>0.94</td>
<td>1.39</td>
<td>2.74</td>
<td>4.36</td>
<td>8.56</td>
<td>31.7</td>
<td>140</td>
</tr>
<tr>
<td>$\sigma[T]$ for PART-TSP</td>
<td>0.48 (70%)</td>
<td>0.77 (65%)</td>
<td>1.22 (130%)</td>
<td>2.11 (152%)</td>
<td>3.27 (119%)</td>
<td>5.35 (123%)</td>
<td>13.3 (155%)</td>
<td>27.1 (85%)</td>
<td>101 (72%)</td>
</tr>
<tr>
<td>$\sigma[T]$ for NN</td>
<td>0.47 (68%)</td>
<td>0.76 (64%)</td>
<td>1.26 (134%)</td>
<td>2.14 (154%)</td>
<td>3.57 (130%)</td>
<td>6.18 (142%)</td>
<td>12.5 (146%)</td>
<td>31.1 (98%)</td>
<td>147 (105%)</td>
</tr>
</tbody>
</table>
### Table 3.2
Comparison of PART-n-TSP, PART-TSP and Nearest Neighbor on $E[T]$ and $\sigma[T]$

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>PART-n-TSP</th>
<th>PART-TSP</th>
<th>Nearest Neighbor</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.02</td>
<td>1.16</td>
<td>1.16 (97%)</td>
</tr>
<tr>
<td>0.2</td>
<td>1.10 (126%)</td>
<td>1.37 (91%)</td>
<td>1.36 (90%)</td>
</tr>
<tr>
<td>0.3</td>
<td>1.22</td>
<td>1.71 (80%)</td>
<td>1.66 (77%)</td>
</tr>
<tr>
<td>0.4</td>
<td>1.59</td>
<td>2.35 (105%)</td>
<td>2.16 (97%)</td>
</tr>
<tr>
<td>0.5</td>
<td>2.77</td>
<td>3.63 (123%)</td>
<td>2.93 (100%)</td>
</tr>
<tr>
<td>0.6</td>
<td>4.10</td>
<td>6.24 (126%)</td>
<td>4.50 (91%)</td>
</tr>
<tr>
<td>0.7</td>
<td>5.67</td>
<td>8.10 (75%)</td>
<td>8.10 (75%)</td>
</tr>
<tr>
<td>0.8</td>
<td>7.82</td>
<td>10.60 (68%)</td>
<td>18.0 (68%)</td>
</tr>
<tr>
<td>0.9</td>
<td>10.42</td>
<td>11.93 (49%)</td>
<td>78.7 (41%)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sigma[T]$</th>
<th>PART-n-TSP</th>
<th>PART-TSP</th>
<th>Nearest Neighbor</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.54 (78%)</td>
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<td>0.54 (78%)</td>
<td>0.54 (78%)</td>
</tr>
<tr>
<td>0.75 (82%)</td>
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<td>0.75 (82%)</td>
<td>0.75 (82%)</td>
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<tr>
<td>1.07 (67%)</td>
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<td>1.07 (67%)</td>
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<tr>
<td>1.59 (126%)</td>
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<td>1.59 (126%)</td>
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<tr>
<td>2.23 (172%)</td>
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<tr>
<td>3.37 (218%)</td>
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<tr>
<td>5.67 (254%)</td>
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<td>5.67 (254%)</td>
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<tr>
<td>8.10 (392%)</td>
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<td>8.10 (392%)</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$E[T]$</th>
<th>PART-n-TSP</th>
<th>PART-TSP</th>
<th>Nearest Neighbor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.02</td>
<td>1.16</td>
<td>1.16</td>
<td>1.16 (97%)</td>
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<tr>
<td>1.10</td>
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<td>1.36</td>
<td>1.36 (90%)</td>
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<tr>
<td>1.22</td>
<td>1.71</td>
<td>1.66</td>
<td>1.66 (77%)</td>
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<tr>
<td>1.59</td>
<td>2.35</td>
<td>2.16</td>
<td>2.16 (97%)</td>
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<td>2.93 (100%)</td>
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<td>4.10</td>
<td>6.24</td>
<td>4.50</td>
<td>4.50 (91%)</td>
</tr>
<tr>
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<td>8.10</td>
<td>8.10 (75%)</td>
</tr>
<tr>
<td>7.82</td>
<td>10.60</td>
<td>10.60</td>
<td>10.60 (68%)</td>
</tr>
<tr>
<td>10.42</td>
<td>11.93</td>
<td>11.93</td>
<td>11.93 (49%)</td>
</tr>
<tr>
<td>15.67</td>
<td>15.15</td>
<td>15.15</td>
<td>15.15 (33%)</td>
</tr>
<tr>
<td>20.82</td>
<td>19.30</td>
<td>19.30</td>
<td>19.30 (27%)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sigma[T]$</th>
<th>PART-n-TSP</th>
<th>PART-TSP</th>
<th>Nearest Neighbor</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.54</td>
<td>0.54</td>
<td>0.54</td>
<td>0.54 (78%)</td>
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<tr>
<td>0.75</td>
<td>0.75</td>
<td>0.75</td>
<td>0.75 (82%)</td>
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<tr>
<td>1.07</td>
<td>1.07</td>
<td>1.07</td>
<td>1.07 (67%)</td>
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<tr>
<td>1.59</td>
<td>1.59</td>
<td>1.59</td>
<td>1.59 (126%)</td>
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<tr>
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<td>2.23</td>
<td>2.23</td>
<td>2.23 (172%)</td>
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<tr>
<td>3.37</td>
<td>3.37</td>
<td>3.37</td>
<td>3.37 (218%)</td>
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<tr>
<td>5.67</td>
<td>5.67</td>
<td>5.67</td>
<td>5.67 (254%)</td>
</tr>
<tr>
<td>8.10</td>
<td>8.10</td>
<td>8.10</td>
<td>8.10 (392%)</td>
</tr>
</tbody>
</table>

Note: $\rho$ and $\sigma[T]$ values are based on simulations.
arrive at a partition, and find the number of tasks to be less than \( n \). Then the vehicle would switch to the next partition without serving any task, resulting in a switching cost but no tasks served.

PART-\( n \)-TSP behaves like a “standardized” version of PART-TSP. While the fixed \( n \) reduces flexibility, it increases certainty. Thus \( Var[T] \) or \( \sigma[T] \) should be lower. Indeed, as shown in Tables 3.1 and 3.2, PART-\( n \)-TSP achieves lower \( \sigma[T] \) than PART-TSP and NN when \( \rho \) is not too small or too large, e.g. when \( \rho \in \{0.3, \ldots, 0.7\} \) for \( B_i \sim [0,0.5] \), and when \( \rho \in \{0.4, \ldots, 0.8\} \) for \( B_i \sim [0,1] \). The performance of PART-\( n \)-TSP on \( \sigma[T] \) when \( \rho \) is too small or too large is not as good for the same reasons affecting \( E[T] \) as explained in the previous paragraph.

### 3.4. Optimality of PART-\( n \)-TSP under light and heavy loads.

The PART-\( n \)-TSP can be modified to yield asymptotically optimal \( E[T] \) under light load \((\rho \to 0^+)\). First the PART-\( n \)-TSP becomes FCFS policy when setting \( r = 1 \) and \( n = 1 \). Then under FCFS policy, let the vehicle return to the median of region \( A \) when it becomes idle. Under light load this is the stochastic queue median (SQM) policy [4], where the vehicle travels directly to the task location from the median, executes the task, and then returns to the median after completion. SQM is proven to be \( E[T] \) optimal under light load [4], proving the optimality of PART-\( n \)-TSP under light load.

Under heavy load \((\rho \to 1^-)\), the following lower bound holds [5].

\[
E[T] \geq \frac{\beta^2_{SP,2\lambda}}{2v^2(1-\rho)^2} \left( \int_A f_k^2(x) \, dx \right)^2
\]

The following theorem shows that PART-\( n \)-TSP achieves the heavy-load lower bound (3.29) when \( r \to \infty \). Thus PART-\( n \)-TSP is asymptotically optimal in \( E[T] \) under heavy load.

**Theorem 3.2.** Under PART-\( n \)-TSP as per Definition 3.1, when \( \rho \to 1^- \) and

---

**Figure 3.6.** Part of simulated data for PART-TSP: \( \rho = 0.9 \), \( B_i \sim [0,0.5] \), and \( r = 25 \).
$n \to \infty$, the system time for the 1-DTRP satisfies

$$E[T] \leq \left(1 + \frac{1}{r}\right) \frac{\beta_{TSP,2}^2}{2v^2(1 - \rho)^2}$$

where $r$ is the number of partitions.


And by (3.4)

$$E[W_O] = \frac{(n-1)r}{2\lambda} < \frac{nr}{2\lambda}$$

By (3.6) and (3.8), and conditioning on the position that a given task takes within its set, and noting that the travel time around the TSP path is no more than the length of the path itself, the expected wait for completion once a task’s set enters service

$$E[W_I] \leq \frac{1}{v}E[D_n] + \frac{1}{n} \sum_{j=1}^{n} jb = \frac{1}{v}E[D_n] + \frac{n+1}{2}b$$

Given that a demand falls in partition $A^k$, the conditional density for its location (whose support is $A^k$) is $f_{A^k}(x) dx$. From [24] we know that, almost surely,

$$\lim_{n \to \infty} \frac{D_n}{\sqrt{n}} = \beta_{TSP,2} \int_{A^k} \sqrt{f_{A^k}(x) dx},$$

where $\beta_{TSP,2}$ is a constant. Let $C = \frac{1}{v} \beta_{TSP,2} \int_{A^k} \sqrt{f_{A^k}(x) dx}$, thus $C$ is a constant. So $\lim_{n \to \infty} \frac{1}{v}E[D_n] = C\sqrt{n}$.

The load of a set $\rho^* = \lambda^* E[W_n] = \frac{\lambda}{n} \left(\frac{E[D_n]}{v} + nb\right) = \lambda b + \lambda^* E[D_n] = \rho + \lambda^* \frac{E[D_n]}{nv}$ by (3.11), (3.12) and (3.20). Thus $\lim_{n \to \infty} \rho^* = \rho + \lambda \lim_{n \to \infty} \frac{E[D_n]}{nv} = \rho + \lambda \lim_{n \to \infty} \frac{C}{n\sqrt{n}} = \rho$. So $\rho \to 1^-$ implies $\rho^* \to 1^-$ when $n \to \infty$.

As for $E[W_P]$, from [28] we know that the mean waiting time in a polling system with renewal arrivals as $\rho^* \to 1^-$ is

$$E[W_P] = \frac{\omega}{1 - \rho^*} + o\left((1 - \rho^*)^{-1}\right),$$

where $\omega = \frac{1 - \rho^*}{2} \left(\frac{\sigma^2}{\sum_{k=1}^{n} \rho^k(1-\rho^k)} + E[\Delta]\right)$ under the exhaustive policy, and $\omega = \frac{1 + \rho^*}{2} \left(\frac{\sigma^2}{\sum_{k=1}^{n} \rho^k(1+\rho^k)} + E[\Delta]\right)$ under the gated policy. Substituting (3.16) we have

$\omega = \frac{\sigma^2}{2} + \frac{\tau + 1}{2\tau} E[\Delta]$ under the exhaustive policy, and $\omega = \frac{\sigma^2}{2} + \frac{\tau + 1}{\tau} E[\Delta]$ under the gated policy. $\Delta = \sum_{k=1}^{n} \Delta^k$ is the total switch time of a polling cycle. $\Delta$ does not depend on $\rho$, $\rho^*$ or $n$, and is given in (3.1) when the region $A$ and partitions $A^k$ are squares.

Thus $E[W_P] = \frac{1}{(1-\rho^*)} \left(\sigma^2 + \frac{\tau + 1}{\tau} E[\Delta]\right) + o\left((1 - \rho^*)^{-1}\right)$ when $\rho^* \to 1^-$. Let $C' = \frac{\tau + 1}{\tau} E[\Delta]$. So $E[W_P] = \frac{\lambda}{2(1-\rho^*)} \left(\sigma^2 + C'\right)$ when $\rho^* \to 1^-$. Since $r$ is a finite natural number, and $\Delta_k$ is upper bounded by the diameter of region $A$, then $E[\Delta]$ is a positive finite number. Thus, $C'$ is a positive finite number.

By (3.18) $\sigma^2 = \hat{\lambda}^* \left(Var[W_n] + \frac{1}{n\lambda^*}\right)$, where $\hat{\lambda}^* = \frac{\lambda}{n}$ by (3.11). Also, $\rho^* = \hat{\lambda}^* E[W_n]$ as $\rho^* \to 1^-$ by (3.12).
So \( E [W_\rho] = \frac{\lambda \left( \frac{\lambda}{nw} + \frac{\lambda}{nE[D_n]} \right) + C'}{2 (1 - \rho E[D_n])} + C' \)

\[ = \frac{\lambda \left( \frac{\lambda}{nw} + \frac{\lambda}{nE[D_n]} + \frac{\lambda}{n\sigma^2} \right) + C'}{2 (1 - \rho E[D_n])} \]

\[ = \frac{\lambda \left( \frac{\lambda}{nw} + \frac{\lambda}{n\sigma^2} + \frac{\lambda}{n\sigma^2} \right) + C'}{2 (1 - \rho E[D_n])} + C' \]

\[ \text{as } \rho^* \to 1^- \], where we substituted \( \lambda = \frac{1}{n} \) and (3.20) and (3.21) in the first equality.

Since \( \lambda \) is the value of \( \lambda \) when \( \rho^* = 1 \), and \( \rho \to 1^- \) implies \( \rho^* \to 1^- \), then we can write

\[ (3.34) \]

\[ E [W_\rho] = \frac{\lambda \left( \frac{1}{nw} + \frac{\lambda}{n\sigma^2} + \frac{\lambda}{n\sigma^2} \right) + C'}{2 (1 - \rho - \lambda \frac{1}{n} E[D_n])} \]

when \( \rho \to 1^- \), where we substituted \( \rho = \lambda b \).

From [19, p.189] we know \( \lim_{n \to \infty} Var [D_n] = O(1) \), and therefore, \( \lim_{n \to \infty} \frac{Var [D_n]}{n} = 0 \).

Thus when \( \rho \to 1^- \) and \( n \to \infty \),


\[ \leq \frac{(n+1)b}{2 \lambda} + \frac{\lambda}{n} E [D_n] \]

\[ \leq \frac{\lambda}{2 \lambda} + C \sqrt{n} + \frac{h b}{2 (1 - \rho - \lambda \frac{1}{n} E[D_n])} \]

Substituting \( b = \frac{\rho}{\lambda} \) we have

\[ (3.35) \]

\[ E[T] \leq \frac{\lambda \left( \frac{1}{nw} + \frac{\lambda}{n\sigma^2} \right) + C'}{2 (1 - \rho - \lambda \sqrt{n})} + \frac{(r + \rho)}{2 \lambda} + C \sqrt{n} \]

We want to minimize (3.35) with respect to \( n \) to get the least upper bound. Noting that (3.35) is convex with respect to \( n \), so there is indeed a minimum. First, however, consider a change of variable \( y = \frac{\lambda}{(1 - \rho) \sqrt{n}} \). With this change,

\[ (3.36) \]

\[ E[T] \leq \frac{\lambda \left( \frac{1}{nw} + \frac{\lambda}{n\sigma^2} \right) + C'}{2 (1 - \rho - \frac{\lambda}{1 - \rho} \sqrt{y})} + \frac{\lambda C^2}{2 (1 - \rho) \sqrt{y}^2} + \frac{\lambda C^2}{2 (1 - \rho) \sqrt{y}^2} \]

For \( \rho \to 1^- \), one can verify that the optimum \( y \) approaches 1. Linearizing the last two terms above about \( y = 1 \) we have \( \frac{\lambda C^2}{2 (1 - \rho) \sqrt{y}^2} = \frac{\lambda C^2}{2 (1 - \rho) \sqrt{y}^2} (3 - 2 y) \), and \( \frac{\lambda C^2}{2 (1 - \rho) \sqrt{y}^2} = \frac{\lambda C^2}{2 (1 - \rho) \sqrt{y}^2} (2 - y) \).

Thus, \( g(y) = \frac{\lambda \left( \frac{1}{nw} + \frac{\lambda}{n\sigma^2} \right) + C'}{2 (1 - \rho - \frac{\lambda}{1 - \rho} \sqrt{y})} + \frac{\lambda C^2}{2 (1 - \rho) \sqrt{y}^2} (3 - 2 y) \)

\[ \approx \frac{C_1}{1 - y} + C_2 (3 - 2 y) + C_3 (2 - y) \]

\[ = \frac{C_1}{1 - y} + (2 C_2 + C_3) (1 - y) + C_2 + C_3, \] where \( C_1 = \frac{\lambda \left( \frac{1}{nw} + \frac{\lambda}{n\sigma^2} \right) + C'}{2 (1 - \rho - \frac{\lambda}{1 - \rho} \sqrt{y})} \), \( C_2 = \frac{\lambda C^2}{2 (1 - \rho) \sqrt{y}^2} (3 - 2 y) \), and \( C_3 = \frac{\lambda C^2}{2 (1 - \rho) \sqrt{y}^2} \).

The approximation for \( g(y) \) is minimized when \( \frac{C_1}{1 - y} = (2 C_2 + C_3) (1 - y) \). Substituting \( C_1, C_2 \) and \( C_3 \) we have an approximate optimum value

\[ (3.37) \]

\[ y^* = 1 - \frac{1}{C} \left( \frac{\lambda}{n} + \frac{\lambda}{n\sigma^2} \right) \]

\[ \frac{C}{2 (1 + r)} \]
Substituting (3.37) into (3.36) and noting that for $\rho \to 1^-$ the approximate $y^*$ approaches 1 we have

$$E[T] \leq \frac{\lambda C^2(r + 1)}{2(1 - \rho)^2} + \frac{\lambda C \sqrt{2(r + 1) \left( \frac{1}{2} + \sigma_B^2 + \frac{C'}{2} \right)}}{2(1 - \rho)^2} + \frac{\lambda C^2}{1 - \rho}$$

when $\rho \to 1^-$. Thus $E[T] \leq \frac{\lambda C^2(r + 1)}{2(1 - \rho)^2} + o((1 - \rho)^{-2})$ when $\rho \to 1^-$. We have

$$E[T] \leq \frac{\lambda C^2(r + 1)}{2(1 - \rho)^2}$$

when $\rho \to 1^-$. Substituting $C$ in (3.39) we have

$$E[T] \leq \frac{\lambda C^2(r + 1)}{2(1 - \rho)^2}$$

The PART-$n$-TSP is optimal under light load. Moreover, when $r \to \infty$, the PART-$n$-TSP policy achieves the heavy-load lower bound (3.29). Therefore the PART-$n$-TSP is both optimal under light load and arbitrarily close to optimality under heavy load, and stabilizes the system for every load $\rho \in [0, 1)$. Notice that with $r = 10$ the PART-$n$-TSP is already guaranteed to be within 10% of the optimal performance under heavy load.

**4. Summary.** We give a good approximation for the distribution of the system time that is easy to compute under the PART-$n$-TSP policy by utilizing the approximation results of the distribution of system time $T$, together with $E[T]$ and $Var[T]$ in the polling systems [8, 15]. We compare PART-$n$-TSP with PART-TSP [31] and Nearest Neighbor [4] on $E[T]$ and $\sigma[T]$ in Tables 3.1 and 3.2, since the latter two are considered near optimal in the literature. The results show that in practice PART-$n$-TSP achieves lower $\sigma[T]$ than PART-TSP and NN and lower $E[T]$ than PART-TSP when the load $\rho$ is not too small or too large. We also prove that PART-$n$-TSP is $E[T]$ optimal under light load ($\rho \to 0^+$) and asymptotically optimal under heavy load ($\rho \to 1^-$) in Theorem 3.2.

**5. Conclusion.** We prove a necessary and sufficient condition for stability in the Dynamic Traveling Repairman Problem (DTRP) [4] under the class of Polling-Sequencing (P-S) policies satisfying unlimited-polling and economy of scale. The number of tasks inside each polling partition is shown to be a Markov chain. Non-location based policies and some common location based policies such as TSP, NN and DA are shown to have economy of scale. The P-S class includes some of the policies proven to be optimal for the expectation of system time under light and heavy loads in the DTRP literature. We give a close form approximation of the distribution of the steady state system time, together with its expectation and variance for PART-$n$-TSP.
policy in the P-S class, which is proved to be optimal for the expectation of system time under light and heavy loads.

REFERENCES


